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Sufficient conditions in the collision avoidance problem under geometrical and integral limitations on control

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ABSTRACT

The paper considers the collision avoidance problem in a system, described by a controlled equation in partial derivatives, containing the second derivative with respect to time and elliptic operator. New spaces, depending on nonnegative parameter, are formed with the help of generalized eigenvalues and eigenfunctions. It is proved here that in the whole scale of these spaces there is unique solution of this hyperbolic equation with the elliptic operator. At that, the solution and its derivative are continuous in time with respect to the related norm. Sufficient conditions for the collision avoidance in the problems obtained under geometrical and integral limitations on the control parameters were gained.

Key words: optimal control, collision avoidance problem, controlled systems with distributed parameters

INTRODUCTION

It is known that many natural processes and phenomena are described by equations in partial derivatives. Vibrations of limited volumes, mathematical model of which is hyperbolic-type equations with elliptic operator, in particular, refer to such processes. To solve such equations, it is necessary first of all to enlarge the domain of definition of the elliptic operator and the operator itself to a self-adjoined one and then to prove existence of a solution, belonging to the energetic space of this operator. At that it should be noted that to prove the solution existence, the fact that widened operator has generalized eigenvalues and generalized eigen functions, composing complete system both in the operator's energetic space and in each space, is used [1, 2]. Note that the collision avoidance problem is mentioned in [3]. At that, works [4-10] should be noted as priority in this direction. The work [1] offers so called equation decomposition method used in this work also. Note that the above mentioned works mainly consider a pursuit problem – inverse to the collision avoidance problem.

Differential operator A of view [1] is considered in space $L_2(\Omega)$:

$$Az = -\sum_{i,j}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x) \frac{\partial z}{\partial x_j} \right), \qquad x \in \Omega, \quad a_{i,j}(x) = a_{j,i}(x) \in C^1(\overline{\Omega}) \tag{1}$$

where Ω is the bounded by piecewise-smooth boundary domain in $R^n, n \geq 1$. The domain of definition $D(A)$ of the operator A is $\dot{C}^2(\Omega)$ (the space of twice continuously differentiable functions). The coefficients $a_{i,j}(\cdot)$ meet the following condition: there is constant $\gamma \neq 0$ that for all $x \in \Omega$ and $(\xi_1, \xi_2, \ldots, \xi_n) \in R^n$ the following inequality holds

$$\sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \geq \gamma^2 \sum_{i=1}^{n} \xi_i^2 . \tag{2}$$

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Assuming $(z,y)_A=(Az,y),\quad z,y\in\dot{C}^2(\Omega)$, it is possible to show that $(\cdot,\cdot)_A$ meets all requirements of scalar product.

Thus, $\dot{C}^2(\Omega)$ is transformed into a Hilbert space. However, it is incomplete concerning the norm

$$\|z\|_A=(Az,z)^{1/2},\quad z\in\dot{C}^2(\Omega) \tag{3}$$

provided by the scalar product $(\cdot,\cdot)_A$. Completing $\dot{C}^2(\Omega)$ concerning the norm $\|\cdot\|_A$, we obtain the complete Hilbert space, called the energetic space of the operator A.

It is known [1, 2], that under the fulfillment of the condition (2) the operator A has discrete spectrum, more precisely, has infinite sequence of the generalized eigenvalues $0<\lambda_1\le\lambda_2\le\dots$ with limit in the infinity and infinite sequence of the generalized eigenfunctions $\varphi_1,\varphi_2,\dots,$ composing the complete system $\{\varphi_i\}$ in the space $L_2(\Omega)$. We will consider $(\varphi_i,\varphi_j)=\delta_{ij}$, where δ_{ij} is Kronecker symbol.

Let r be arbitrary nonnegative number. Let's introduce a denotation

$$l_r=\{\alpha=(\alpha_1,\alpha_2,\dots):\sum_{i=1}^{\infty}\lambda_i^r\alpha_i^2\}<\infty$$

$$H_r(\Omega)=\left\{f\in L_2(\Omega): f=\sum_{i=1}^{\infty}\alpha_i\varphi_i,\ \alpha\in l_r\right\} \tag{4}$$

Let's define in the spaces l_r, $H_r(\Omega)$ the scalar products and norms:

$$(\alpha,\beta)_r=\sum_{i=1}^{\infty}\lambda_i^r\alpha_i\beta_i,\quad \alpha,\beta\in l_r,\qquad \|\alpha\|=(\alpha,\alpha)_r^{1/2};$$

$$(f,g)_r=(\alpha,\beta)_r,\quad f=\sum_{i=1}^{\infty}\alpha_i\varphi_i,\quad g=\sum_{i=1}^{\infty}\beta_i\varphi_i,\qquad \|f\|=\|g\| \tag{5}$$

Note that $H_0(\Omega)=L_2(\Omega)$ and $H_r(\Omega)\subset H_s(\Omega)$ for arbitrary $0\le s\le r$.

We denote by $C(0,T;H_r(\Omega))(L_2(0,T;H_r(\Omega)))$ a space, consisting of continuous (square integrable measurable) functions, defined on $[0,T]$ and with values in $H_r(\Omega)$, where T is some positive constant.

Statement of the problem. Let's consider the following controlled distributed system:

$$\begin{aligned}&\frac{d^2z(t)}{dt^2}+Az(t)=-u(t)+v(t),\quad 0<t\le T,\\ &u(\cdot),v(\cdot)\in L_2(0,T;H_r(\Omega)),\\ &z(0)=z^{(0)},\quad z^{(0)}\in H_{r+1}(\Omega),\quad \dot{z}(0)=\dot{z}^{(0)},\quad \dot{z}^{(0)}\in H_r(\Omega),\end{aligned} \tag{6}$$

where operator A is defined in the view (1).

In [2] it is found that in the space $C(0,T;H_{r+1}(\Omega))$ there is unique function $z(t),\quad 0\le t\le T$, which is the solution of the problem (6) in the sense of the generalized function theory (distribution theory) and $\dot{z}(t)\in C(0,T;H_r(\Omega))$.

The functions $u(\cdot)$ and $v(\cdot)$ are called controls of the confrontational sides. They satisfy restrictions, defined by one of the following inequality systems:

$$\|u(t)\|\le\rho,\quad \|v(t)\|\le\sigma,\quad 0\le t\le T; \tag{7}$$

$$\|u(\cdot)\| \le \rho, \quad \|v(\cdot)\| \le \sigma; \qquad (8)$$

where ρ and σ are nonnegative constants.

Let's call the controls $u(\cdot)$ and $v(\cdot)$, satisfying one of the conditions (7)-(8), as permitted. Let's call the controlled system (6), in which $u(\cdot)$ and $v(\cdot)$ satisfy the inequalities (7), (8) as the problem ((6),(7)); ((6),(8)).

Definition. We say that in the problem ((6) and (7)); ((6) and (8))) one can avoid the collision from the initial position $z_0 = (z^{(0)}, \dot{z}^{(0)}), z_0 \ne 0$, if for an arbitrary fixed positive number T it is possible to consruct a control $v_0(\cdot)$ such that

1) $\|v_0(t)\| \le \sigma \quad (\|v_0(\cdot)\| \le \sigma$ in ((6),(8));

2) for the arbitrary control $u_0(\cdot)$, satisfying the inequality $\|u_0(t)\| \le \rho \quad (\|u_0(\cdot)\| \le \rho$ in ((6),(8)) the solution $z_0(t), \quad 0 \le t \le T$, of the problem (6), where $u(\cdot) = u_0(\cdot)$, $v(\cdot) = v_0(\cdot)$, and its derivative $\dot{z}_0(t), \quad 0 \le t \le T$, don't simultaneously become zero. At that, in order to find the value $v_0(t)$ of the control $v_0(\cdot)$ at every time moment t it is permitted to use the values:

a) z_0 in ((6),(7));

b) z_0 and $u_0(s)$, $t-\theta \le s < t$ ($u_0(s), 0 \le s \le t$) at $t < \theta$), in ((6),(8)), where θ is an arbitrary positive fixed number.

The collision avoidance problem consists in finding of the initial positions z_0, from which it is possible to avoid the collisions (with the point 0), as well as in the explicit constructing of the control $v_0(\cdot)$.

Theorem. 1) If $\sigma \ge \rho$, then in the problems ((6),(7)) and ((6),(8)) it is possible to avoid collisions from any initial position $z_0, z_0 \ne 0$.

Proof. 1) a) Let's consider the problem ((6),(7)). Let $\sigma \ge \rho$, T is a positive number, $u_0(\cdot)$ is an arbitrary control, $\|u_0(t)\| \le \rho$, z_0 is an arbitrary initial position, $z_0 \ne 0$.

We denote through $v_0(\cdot)$ the arbitrary control for now, its concrete form will be designated later.

Let

$$u_0(t) = \sum_{i=1}^{\infty} u_i(t)\varphi_i, \quad v_0(t) = \sum_{i=1}^{\infty} v_i(t)\varphi_i,$$

$$z^{(0)} = \sum_{i=1}^{\infty} z_i^{(0)}\varphi_i, \quad \dot{z}^{(0)} = \sum_{i=1}^{\infty} \dot{z}_i^{(0)}\varphi_i, \quad z_0(t) = \sum_{i=1}^{\infty} z_i(t)\varphi_i$$

are expansions in Fourier series, the vectors $u_0(t)$, $v_0(t)$, $z^{(0)}$, $\dot{z}^{(0)}$ and $z_0(t)$, and $u_i(t)$, $v_i(t)$, $z_i^{(0)}$, $\dot{z}_i^{(0)}$, $z_i(t)$ are their corresponding Fourier coefficients.

Substituting these expansions in the equation (6) and equating the related coefficients at φ_i, we obtain the infinite system of differential equations of second order

$$\frac{d^2 z_i(t)}{dt^2} + \lambda_i z_i(t) = -u_i(t) + v_i(t), \quad i = 1, 2, \ldots, \qquad (11)$$

and initial conditions $z_i(0) = z_i^{(0)}$, $\dot{z}_i(0) = \dot{z}_i^{(0)}$.

As far as $z_0 \neq 0$, then there are two possible cases: I. $z^{(0)} \neq 0$; II. $z^{(0)} = 0$, $\dot{z}^{(0)} \neq 0$.

Let's consider the case I, because the case II is studied similarly. Let k be the least value of index i, for which $z_k^{(0)} \neq 0$. For the convenience of calculations let's assume $\eta_1 = z_k$, $\eta_2 = \dot{z}_k$. Then

$$\dot{\eta}_1(t) = \eta_2(t),\ \dot{\eta}_2(t) = -\lambda_k \eta_1(t) - u_k(t) + v_k(t),\ 0 \le t \le T,$$

$$\eta_1(0) = z_k^{(0)},\ \eta_2(0) = \dot{z}_k^{(0)} \qquad (12)$$

It is clear that

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = e^{tC}\left\{ \begin{pmatrix} z_k^{(0)} \\ \dot{z}_k^{(0)} \end{pmatrix} + \int_0^t e^{-sC}(-\bar{u}_k(s) + \bar{v}_k(s))ds \right\}, \qquad (13)$$

where

$$C = \begin{pmatrix} 0 & 1 \\ -\lambda_k & 0 \end{pmatrix},\quad e^{tC} = \begin{pmatrix} \cos\sqrt{\lambda_k}t & \frac{1}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t \\ -\sqrt{\lambda_k}\sin\sqrt{\lambda_k}t & \cos\sqrt{\lambda_k}t \end{pmatrix},$$

$$\bar{u}_k = \begin{pmatrix} 0 \\ u_k \end{pmatrix},\qquad \bar{v}_k = \begin{pmatrix} 0 \\ v_k \end{pmatrix}. \qquad (14)$$

Therefore ((13), (14))

$$e^{-tC}\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} z_k^{(0)} \\ \dot{z}_k^{(0)} \end{pmatrix} + \begin{pmatrix} \int_0^t \left\{ -\frac{1}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}\, s[v_k(s) - u_k(s)] \right\} ds \\ \int_0^t \cos\sqrt{\lambda_k}\, s[v_k(s) - u_k(s)]ds \end{pmatrix} \qquad (15)$$

Now it is clear that if $z_0(t) = \dot{z}_0(t) = 0$ at some $t = t' \in [0, T]$, then $z_0(t') = \dot{z}_0(t') = 0$, i.e. $\eta_1(t') = \eta_2(t') = 0$. Hence (15),

$$z_k^{(0)} + \int_0^{t'} \left\{ -\frac{1}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}\, s[v_k(s) - u_k(s)] \right\} ds = 0$$

$$\dot{z}_k^{(0)} + \int_0^{t'} \cos\sqrt{\lambda_k}\, s[v_k(s) - u_k(s)]ds = 0$$

Now let's show that it is possible to choose the functions $v_i(t),\quad 0 \le t \le T$, $i = 1, 2, \ldots,$ so that

$$\delta(t) = z_k^{(0)} - \int_0^t \frac{1}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}\, s[v_k(s) - u_k(s)]ds \neq 0 \qquad (16)$$

on $[0, T]$. It follows therefrom that at such way of control the function $(z_0(t), \dot{z}_0(t))$, $0 \le t \le T$, does not become zero.

In fact, let $v_i(\cdot)=0$ for all $i \neq k$, and $v_k(\cdot)$ choose as follows (considering $z_k^{(0)}>0$; in case when $z_k^{(0)}<0$, the discussions are produced similarly):

$$v_k(t)=-\frac{\sigma}{\sqrt{\lambda_k^r}}\operatorname{sgn}\sin\sqrt{\lambda_k}t, \quad 0 \le t \le T. \qquad (17)$$

Note that at the specified way of choice $v_0(\cdot)$, the inequality $\|v_0(t)\| \le \sigma$ holds obviously.
Further, we have ((16), (17))

$$\delta(t)=z_k^{(0)}+\frac{\sigma}{\sqrt{\lambda_k^{1+r}}}\int_0^t\left|\sin\sqrt{\lambda_k}s\right|ds+\frac{1}{\sqrt{\lambda_k}}\int_0^t\sin\sqrt{\lambda_k}su_k(s)ds \qquad (18)$$

Since $\|u(t)\| \le \rho$, then $|u_k(t)| \le \frac{\rho}{\sqrt{\lambda_k^r}}$ ((4), (5)). It means that

$$\sin\sqrt{\lambda_k}su_k(s) \ge -\left|\sin\sqrt{\lambda_k}s\right||u_k(s)| \ge \frac{-\left|\sin\sqrt{\lambda_k}s\right|\rho}{\sqrt{\lambda_k^r}},$$

and as $\sigma \ge \rho$, then

$$\delta(t) \ge z_k^{(0)}+\frac{\sigma}{\sqrt{\lambda_k^{1+r}}}\int_0^t\left|\sin\sqrt{\lambda_k}s\right|ds-\frac{\rho}{\sqrt{\lambda_k^{1+r}}}\int_0^t\left|\sin\sqrt{\lambda_k}s\right|ds \ge z_k^{(0)}. \qquad (19)$$

Therefore, for all $t \in [0,T]$ we have

$$\delta(t) \ge z_k^{(0)}>0 \qquad (20)$$

As noted above, it follows from (20) that it is possible to avoid the collision from the initial position z_0.

b) And now let's consider the problem ((6), (8)), considering $\rho \le \sigma$. Arguing as above we obtain the formula (15) and here we come to the conclusion that if $\delta(t) \neq 0$ for all $t \in [0,T]$, then the function $(z_0(t),\dot{z}_0(t))$, $0 \le t \le T$, does not become 0. Thus, it all comes to the fact that by choosing the function $v_k(\cdot)$ you need to achieve the fulfillment of the condition: $\delta(t) \neq 0$ on $[0,T]$.

Let $z_k^{(0)}=3\varepsilon$. Let's consider $v_i(\cdot)=0$ for all $i \neq k$, and construct the function $v_k(\cdot)$ as follows. Setting $v_k(t)=0$ on $[0,\delta]$, $0 \le \delta \le \theta$, $v_k(t)=v_k(t-\delta)$ on $[\delta,T]$, the constant δ will be chosen below ((23), (29)).

Let in the beginning $t \in [0,\delta]$. Then from (16)

$$\delta(t)=z_k^{(0)}-\frac{1}{\sqrt{\lambda_k}}\int_0^t\sin\sqrt{\lambda_k}su_k(s)ds \ge 3\varepsilon-\frac{1}{\sqrt{\lambda_k}}\int_0^t|u_k(s)|ds \qquad (21)$$

and by virtue of the inequality Cauchy-Bunyakovsky

$$\int_0^t|u_k(s)|ds \le \sqrt{t}\sqrt{\int_0^t u_k^2(s)ds} \le \sqrt{\delta}\sqrt{\int_0^t u_k^2(s)ds} \le \sqrt{\delta}\frac{\rho}{\sqrt{\lambda_k^r}}, \qquad (22)$$

under the fulfilment of the inequality

$$\sqrt{\delta}\,\frac{\rho}{\sqrt{\lambda_k^{r+1}}} \le \varepsilon \qquad (23)$$

on the segment $[0,\delta]$ we have $\delta(t) \ge 2\varepsilon$ ((21)-(23)).

Thus, $z_k(t) \ne 0$ on $[0,\delta]$. Let now $t \in [\delta, T]$. Then from (16)

$$\delta(t) = 3\varepsilon + \int_0^{\delta} \frac{1}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\, s\, u_k(s)ds + \int_{\delta}^{t} \frac{1}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\, s\, u_k(s)ds -$$

$$- \int_{\delta}^{t} \frac{1}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\, s\, u_k(s-\delta)ds \qquad (24)$$

It is easy to verify that

$$\int_{\delta}^{t} \frac{1}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\, s\,[u_k(s) - u_k(s-\delta)]ds + \int_{t-\delta}^{t} \frac{1}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\, s\, u_k(s)ds -$$

$$- \int_0^{\delta} \frac{1}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\, s\, u_k(s)ds + \int_{\delta}^{t} \frac{1}{\sqrt{\lambda_k}} \left[\sin\sqrt{\lambda_k}\,(s-\delta) - \sin\sqrt{\lambda_k}\, s\right] u_k(s-\delta)ds \qquad (25)$$

Therefore ((24), (25))

$$\delta(t) = 3\varepsilon + \int_{t-\delta}^{t} \frac{1}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\, s\, u_k(s)ds -$$

$$- \int_{\delta}^{t} \frac{1}{\sqrt{\lambda_k}} \left[\sin\sqrt{\lambda_k}\, s - \sin\sqrt{\lambda_k}\,(s-\delta)\right] u_k(s-\delta)ds \qquad (26)$$

Similarly (22) we have

$$\left| \int_{t-\delta}^{t} \frac{1}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\, s\, u_k(s)ds \right| \le \frac{1}{\sqrt{\lambda_k}} \sqrt{\delta}\, \frac{\rho}{\sqrt{\lambda_k^{r}}} \qquad (27)$$

Further it is obvious that

$$\frac{1}{\sqrt{\lambda_k}} \left| \sin\sqrt{\lambda_k}\, s - \sin\sqrt{\lambda_k}\,(s-\delta) \right| \le \delta \qquad (28)$$

Therefore, if consider the inequalities

$$\sqrt{\delta}\,\frac{\rho}{\sqrt{\lambda_k^{r+1}}} \le \varepsilon, \quad \sqrt{T-\delta}\,\delta\,\frac{\rho}{\sqrt{\lambda_k^{r}}} \le \varepsilon, \qquad (29)$$

as fulfilled, then for any $t \in [\delta, T]$ we obtain ((23), (26)-(29)) $\delta(t) \ge \varepsilon$.

Thus, if choose the function $v_k(\cdot)$ as mentioned above, then on $[0,T]$ the function $\delta(t) \ge \varepsilon$. As noted above it follows from here that it is possible to avoid the collision in the problem (6), (8) from the initial position z_0. The theorem is proved.

CONCLUSIONS

This work considers the collision avoidance problems. At that, various limitations are laid on the control parameters, included in the first part of the equation. The sufficient conditions, providing avoidance of the collisions from all initial positions were gained from the received problems with geometrical ((6), (7)) and integral limits ((6), (8)).

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